Truncation of Spherical Harmonic Series and its Influence on Gravity Field Modelling

T. Fecher , T. Gruber, R. Rummel

Institut für Astronomische und Physikalische Geodäsie, Technische Universität München, Germany, ***e-mail: fecher@bv.tum.de***

Introduction

Least squares adjustment is the classical method of global gravity field modelling in terms of a spherical harmonic series. Since the gravity field is a continuous field function its series representation extends theoretically up to infinity. In reality data are given at discrete points either along satellite tracks or - as discussed here - on a global rectangular grid. As a consequence, the series has to be truncated and one is faced with two fundamental effects: (1) aliasing of high frequency data content into the estimated spherical harmonic coefficients, (2) loss of orthogonality, if the data grid deviates from a Gauss grid. Both effects are discussed here.

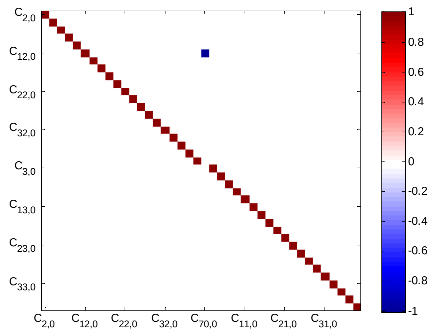

Fig 1: Elements of Matrix $(A^TPA)^{-1}A^TPA^*$ for order 0 (Gaussian grid and weights case). The column $C_{70,0}$ concerning N^{max}=70 causes aliasing at degree 12.

The adjustment model is $c=(A^TPA)^{-1}A^TP\ell$ (1) with c the spherical harmonic coefficients and ℓ the data at grid points. In our experiment ℓ is generated by spherical harmonic synthesis $\ell=A^*c^*$ (2), where c^* is equal to c plus some chosen spherical harmonic coefficients. The maximum resolvable degree $N^{Nyquist}$ depends on the grid spacing in latitude and longitude direction. Insertion of (2) into (1) leads to $c=(A^TPA)^{-1}A^TPA^*c^*$. Here $(A^TPA)^{-1}A^TPA^*$ is a matrix with the dimension $length(c)$ x $length(c^*)$, defining the projector, with which the coefficients c^* are mapped onto c. N^{max} is the highest frequency outside the parameter space.

Experiment 1 (based on a Gaussian grid and weights): $N^{Nyquist}$=41, $c=[C_{0,0}, C_{1,0}, C_{1,1}, S_{1,1}, \ldots, C_{40,40}, S_{40,40}]$, $c^*=[c, C_{70,0}, \ldots, S_{70,70}]$; from c^* follows N^{max}=70;

- **Fig 1** gives the elements of $(A^TPA)^{-1}A^TPA^*$ for order 0, the column for $C_{70,0}$ shows how $C_{70,0}$ is mapped onto the parameter space.
- **Fig 2** is a plot of column $C_{70,0}$; it can be seen that the projections are 0 for elements smaller than degree 12. Degree 12 can be called N^{alias} and can be derived in analogy to 1D FFT from $N^{alias}=2\,N^{Nyquist}-N^{max}$.
- **Fig 3** displays the effect of $(A^TPA)^{-1}A^TPA^*$, i.e. the projections of the coefficients of N^{max}=70 on all coefficients up to $N^{Nyquist}$. It is confirmed that the parameter space smaller than N^{alias} is not affected. For elements larger than N^{alias} there exist projections, because the Gaussian grid and weights only conserves orthogonality for Legendre polynomials in the discrete case if the degree of the polynomial respectively the polynomial product is smaller or equal to $2\,N^{Nyquist}-1$.

Experiment 2 (based on an equidistant grid and constant weights): $N^{Nyquist}$, c and c^* like in Experiment 1.

- **Fig 4** corresponds to Fig 2. Again $C_{12,0}$ is the coefficient mostly affected, but its projection is smaller now, because of the loss of orthogonality of the Legendre polynomials.
- **Fig 5** corresponds to Fig 3. The loss of orthogonality leads to projections not only at degrees larger than $C_{12,0}$, but also at the smaller degrees.

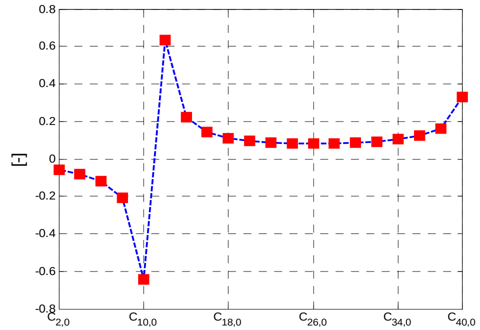

Fig 4: Analogous to Fig 2 for an equidistant grid and constant weighting.

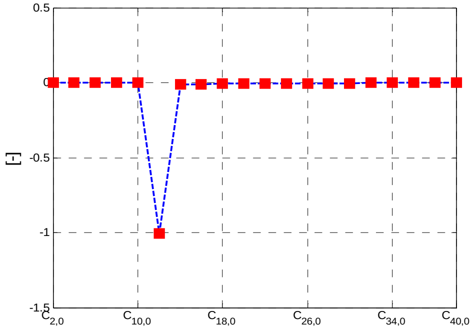

Fig 2: Values of column $C_{70,0}$ of Fig 1. Multiplication of these elements with $C_{70,0}$ describe the aliasing effect caused by $C_{70,0}$.

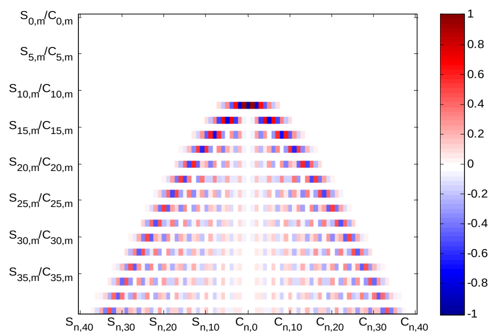

Fig 3: The columns of $(A^TPA)^{-1}A^TPA^*$ for N^{max}=70 can be ordered in triangle form. All elements smaller than N^{alias}=12 are zero; they are not affected by aliasing.

Conclusions

- Using a Gaussian grid and Gaussian weights, all coefficients smaller than $N^{alias}=2\,N^{Nyquist}-N^{max}$ can be calculated free of aliasing effects.
- Using an equidistant grid the coefficients smaller than N^{alias} are free of aliasing errors, but an error is added, which follows from the non-orthogonality of Legendre polynomials in the discrete case.

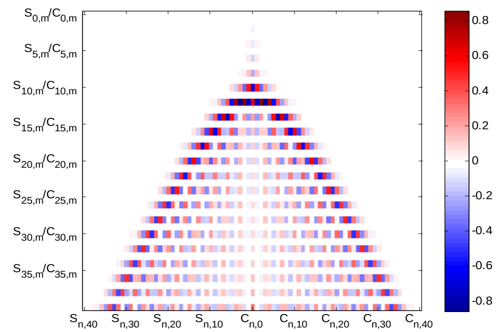

Fig 5: Analogous to Fig 3. In the small degrees an error appears because of the non-orthogonality of the Legendre polynomials.